Chapter 11

THREE DIMENSIONAL GEOMETRY

❖ *The moving power of mathematical invention is not reasoning but imagination. – A. DEMORGAN* ❖

11.1 Introduction

Leonhard Euler (1707-1783)

In Class XI, while studying Analytical Geometry in two dimensions, and the introduction to three dimensional geometry, we confined to the Cartesian methods only. In the previous chapter of this book, we have studied some basic concepts of vectors. We will now use vector algebra to three dimensional geometry. The purpose of this approach to 3-dimensional geometry is that it makes the study simple and elegant*.

In this chapter, we shall study the direction cosines and direction ratios of a line joining two points and also discuss about the equations of lines and planes in space under different conditions, angle between two lines, two planes, a line and a plane, shortest distance between two skew lines and distance of a point from a plane. Most of the above results are obtained in vector form. Nevertheless, we shall also translate these results in the Cartesian form which, at times, presents a more clear geometric and analytic picture of the situation.

11.2 Direction Cosines and Direction Ratios of a Line

From Chapter 10, recall that if a directed line L passing through the origin makes angles α, β and γ with x, y and z-axes, respectively, called direction angles, then cosine of these angles, namely, $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called direction cosines of the directed line L.

If we reverse the direction of L, then the direction angles are replaced by their supplements, i.e., $\pi-\alpha$, $\pi-\beta$ and $\pi-\gamma$. Thus, the signs of the direction cosines are reversed.

* For various activities in three dimensional geometry, one may refer to the Book ***"A Hand Book for designing Mathematics Laboratory in Schools"***, NCERT, 2005

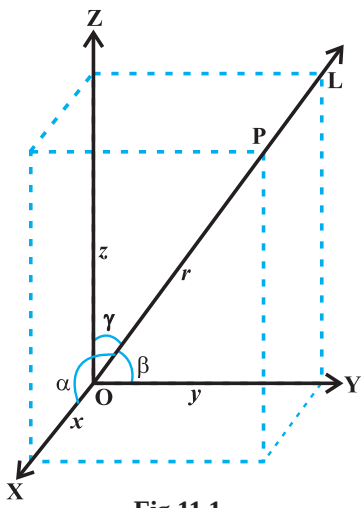

Fig 11.1

Note that a given line in space can be extended in two opposite directions and so it has two sets of direction cosines. In order to have a unique set of direction cosines for a given line in space, we must take the given line as a directed line. These unique direction cosines are denoted by l, m and n.

Remark If the given line in space does not pass through the origin, then, in order to find its direction cosines, we draw a line through the origin and parallel to the given line. Now take one of the directed lines from the origin and find its direction cosines as two parallel line have same set of direction cosines.

Any three numbers which are proportional to the direction cosines of a line are called the *direction ratios* of the line. If l, m, n are direction cosines and a, b, c are direction ratios of a line, then $a = \lambda l$, $b=\lambda m$ and $c = \lambda n$, for any nonzero $\lambda \in \mathbf{R}$.

Note Some authors also call direction ratios as direction numbers.

Let a, b, c be direction ratios of a line and let l, m and n be the direction cosines ($d.c$'s) of the line. Then

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k \text{ (say), } k \text{ being a constant.}$$

Therefore $\quad l = ak,\ m = bk,\ n = ck \quad \ldots (1)$

But $\quad l^2 + m^2 + n^2 = 1$

Therefore $\quad k^2 (a^2 + b^2 + c^2) = 1$

or $$k = \pm \frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

Hence, from (1), the *d.c.'s* of the line are

$$l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

where, depending on the desired sign of k, either a positive or a negative sign is to be taken for l, m and n.

For any line, if a, b, c are direction ratios of a line, then ka, kb, kc; $k \neq 0$ is also a set of direction ratios. So, any two sets of direction ratios of a line are also proportional. Also, for any line there are infinitely many sets of direction ratios.

11.2.1 *Relation between the direction cosines of a line*

Consider a line RS with direction cosines l, m, n. Through the origin draw a line parallel to the given line and take a point P(x, y, z) on this line. From P draw a perpendicular PA on the x-axis (Fig. 11.2).

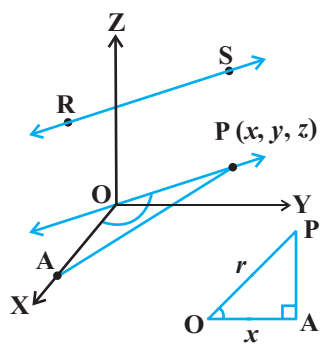

Fig 11.2

Let OP = r. Then $\cos\alpha = \frac{OA}{OP} = \frac{x}{r}$. This gives $x = lr$.

Similarly, $y = mr$ and $z = nr$

Thus $x^2 + y^2 + z^2 = r^2 (l^2 + m^2 + n^2)$

But $x^2 + y^2 + z^2 = r^2$

Hence $\boldsymbol{l^2 + m^2 + n^2 = 1}$

11.2.2 *Direction cosines of a line passing through two points*

Since one and only one line passes through two given points, we can determine the direction cosines of a line passing through the given points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) as follows (Fig 11.3 (a)).

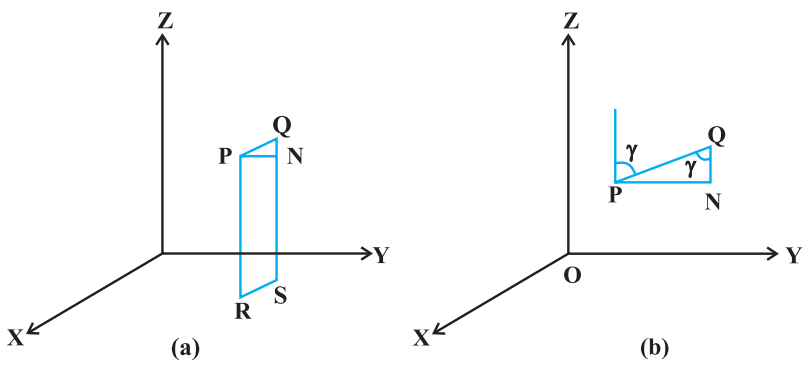

Fig 11.3

Let l, m, n be the direction cosines of the line PQ and let it makes angles α, β and γ with the x, y and z-axis, respectively.

Draw perpendiculars from P and Q to XY-plane to meet at R and S. Draw a perpendicular from P to QS to meet at N. Now, in right angle triangle PNQ, $\angle PQN = \gamma$ (Fig 11.3 (b).

Therefore, $$\cos\gamma = \frac{NQ}{PQ} = \frac{z_2 - z_1}{PQ}$$

Similarly $$\cos\alpha = \frac{x_2 - x_1}{PQ} \text{ and } \cos\beta = \frac{y_2 - y_1}{PQ}$$

Hence, the direction cosines of the line segment joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are

$$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$$

where $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Note The direction ratios of the line segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ may be taken as

$$x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \text{ or } x_1 - x_2,\ y_1 - y_2,\ z_1 - z_2$$

Example 1 If a line makes angle 90°, 60° and 30° with the positive direction of x, y and z-axis respectively, find its direction cosines.

Solution Let the $d.c.$'s of the lines be l, m, n. Then $l = \cos 90^0 = 0$, $m = \cos 60^0 = \frac{1}{2}$, $n = \cos 30^0 = \frac{\sqrt{3}}{2}$.

Example 2 If a line has direction ratios 2, – 1, – 2, determine its direction cosines.

Solution Direction cosines are

$$\frac{2}{\sqrt{2^2 + (-1)^2 + (-2)^2}}, \frac{-1}{\sqrt{2^2 + (-1)^2 + (-2)^2}}, \frac{-2}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$$

or $$\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$$

Example 3 Find the direction cosines of the line passing through the two points (– 2, 4, – 5) and (1, 2, 3).

Solution We know the direction cosines of the line passing through two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are given by

$$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$$

where $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Here P is (– 2, 4, – 5) and Q is (1, 2, 3).

So $$PQ = \sqrt{(1-(-2))^2 + (2-4)^2 + (3-(-5))^2} = \sqrt{77}$$

Thus, the direction cosines of the line joining two points is

$$\frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}$$

Example 4 Find the direction cosines of x, y and z-axis.

Solution The x-axis makes angles 0°, 90° and 90° respectively with x, y and z-axis. Therefore, the direction cosines of x-axis are cos 0°, cos 90°, cos 90° i.e., 1,0,0. Similarly, direction cosines of y-axis and z-axis are 0, 1, 0 and 0, 0, 1 respectively.

Example 5 Show that the points A (2, 3, – 4), B (1, – 2, 3) and C (3, 8, – 11) are collinear.

Solution Direction ratios of line joining A and B are

1 – 2, – 2 – 3, 3 + 4 i.e., – 1, – 5, 7.

The direction ratios of line joining B and C are

3 –1, 8 + 2, – 11 – 3, i.e., 2, 10, – 14.

It is clear that direction ratios of AB and BC are proportional, hence, AB is parallel to BC. But point B is common to both AB and BC. Therefore, A, B, C are collinear points.

EXERCISE 11.1

1. If a line makes angles 90°, 135°, 45° with the x, y and z-axes respectively, find its direction cosines.
2. Find the direction cosines of a line which makes equal angles with the coordinate axes.
3. If a line has the direction ratios –18, 12, – 4, then what are its direction cosines ?
4. Show that the points (2, 3, 4), (– 1, – 2, 1), (5, 8, 7) are collinear.
5. Find the direction cosines of the sides of the triangle whose vertices are (3, 5, – 4), (– 1, 1, 2) and (– 5, – 5, – 2).

11.3 Equation of a Line in Space

We have studied equation of lines in two dimensions in Class XI, we shall now study the vector and cartesian equations of a line in space.

A line is uniquely determined if

(i) it passes through a given point and has given direction, or

(ii) it passes through two given points.

11.3.1 *Equation of a line through a given point and parallel to a given vector* $\vec{b}$

Let $\vec{a}$ be the position vector of the given point A with respect to the origin O of the rectangular coordinate system. Let l be the line which passes through the point A and is parallel to a given vector $\vec{b}$. Let $\vec{r}$ be the position vector of an arbitrary point P on the line (Fig 11.4).

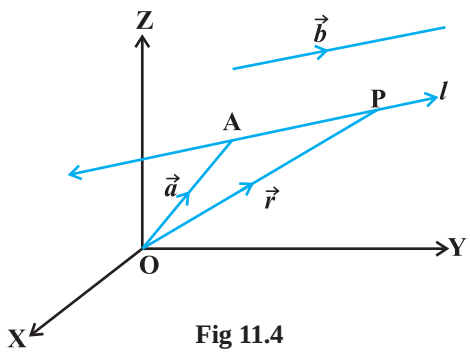

Fig 11.4

Then $\overrightarrow{AP}$ is parallel to the vector $\vec{b}$, i.e., $\overrightarrow{AP} = \lambda\vec{b}$, where λ is some real number.

But $$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$$

i.e. $$\lambda\vec{b} = \vec{r} - \vec{a}$$

Conversely, for each value of the parameter λ, this equation gives the position vector of a point P on the line. Hence, the vector equation of the line is given by

$$\vec{r} = \vec{a} + \lambda\,\vec{b} \qquad \text{... (1)}$$

Remark If $\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$, then a, b, c are direction ratios of the line and conversely, if a, b, c are direction ratios of a line, then $\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$ will be the parallel to the line. Here, b should not be confused with $|\vec{b}|$.

Derivation of cartesian form from vector form

Let the coordinates of the given point A be (x_1, y_1, z_1) and the direction ratios of the line be a, b, c. Consider the coordinates of any point P be (x, y, z). Then

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}\,;\ \vec{a} = x_1\,\hat{i} + y_1\,\hat{j} + z_1\,\hat{k}$$

and $$\vec{b} = a\,\hat{i} + b\,\hat{j} + c\,\hat{k}$$

Substituting these values in (1) and equating the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$x = x_1 + \lambda a;\ \ y = y_1 + \lambda\, b;\ \ z = z_1 + \lambda\, c \qquad \text{... (2)}$$

These are parametric equations of the line. Eliminating the parameter λ from (2), we get

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \qquad \text{... (3)}$$

This is the Cartesian equation of the line.

Note If l, m, n are the direction cosines of the line, the equation of the line is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Example 6 Find the vector and the Cartesian equations of the line through the point (5, 2, – 4) and which is parallel to the vector $3\hat{i} + 2\hat{j} - 8\hat{k}$.

Solution We have

$$\vec{a} = 5\hat{i} + 2\hat{j} - 4\hat{k} \text{ and } \vec{b} = 3\hat{i} + 2\hat{j} - 8\hat{k}$$

Therefore, the vector equation of the line is

$$\vec{r} = 5\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k})$$

Now, $\vec{r}$ is the position vector of any point P(x, y, z) on the line.

Therefore, $$x\hat{i} + y\hat{j} + z\hat{k} = 5\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k})$$

$$= (5 + 3\lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (-4 - 8\lambda)\hat{k}$$

Eliminating λ, we get

$$\frac{x-5}{3} = \frac{y-2}{2} = \frac{z+4}{-8}$$

which is the equation of the line in Cartesian form.

11.3.2 *Equation of a line passing through two given points*

Let $\vec{a}$ and $\vec{b}$ be the position vectors of two points A(x_1, y_1, z_1) and B(x_2, y_2, z_2), respectively that are lying on a line (Fig 11.5).

Let $\vec{r}$ be the position vector of an arbitrary point P(x, y, z), then P is a point on the line if and only if $\overrightarrow{AP} = \vec{r} - \vec{a}$ and $\overrightarrow{AB} = \vec{b} - \vec{a}$ are collinear vectors. Therefore, P is on the line if and only if

$$\vec{r} - \vec{a} = \lambda(\vec{b} - \vec{a})$$

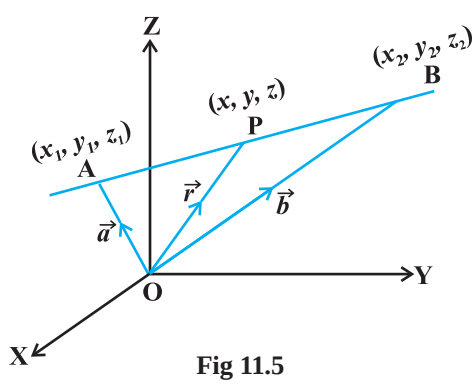

Fig 11.5

or $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$, $\lambda \in \mathbf{R}$. ... (1)

This is the vector equation of the line.

Derivation of cartesian form from vector form

We have

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k},\ \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} \text{ and } \vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k},$$

Substituting these values in (1), we get

$$x\hat{i} + y\hat{j} + z\hat{k} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} + \lambda[(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}]$$

Equating the like coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$, we get

$$x = x_1 + \lambda(x_2 - x_1);\ y = y_1 + \lambda(y_2 - y_1);\ z = z_1 + \lambda(z_2 - z_1)$$

On eliminating λ, we obtain

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

which is the equation of the line in Cartesian form.

Example 7 Find the vector equation for the line passing through the points (–1, 0, 2) and (3, 4, 6).

Solution Let $\vec{a}$ and $\vec{b}$ be the position vectors of the point A(– 1, 0, 2) and B(3, 4, 6).

Then $\vec{a} = -\hat{i} + 2\hat{k}$

and $\vec{b} = 3\hat{i} + 4\hat{j} + 6\hat{k}$

Therefore $\vec{b} - \vec{a} = 4\hat{i} + 4\hat{j} + 4\hat{k}$

Let $\vec{r}$ be the position vector of any point on the line. Then the vector equation of the line is

$$\vec{r} = -\hat{i} + 2\hat{k} + \lambda(4\hat{i} + 4\hat{j} + 4\hat{k})$$

Example 8 The Cartesian equation of a line is

$$\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2}$$

Find the vector equation for the line.

Solution Comparing the given equation with the standard form

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

We observe that $x_1 = -3$, $y_1 = 5$, $z_1 = -6$; $a = 2$, $b = 4$, $c = 2$.

Thus, the required line passes through the point (– 3, 5, – 6) and is parallel to the vector $2\hat{i}+4\hat{j}+2\hat{k}$. Let $\vec{r}$ be the position vector of any point on the line, then the vector equation of the line is given by

$$\vec{r}=(-3\hat{i}+5\hat{j}-6\hat{k})+\lambda\ (2\hat{i}+4\hat{j}+2\hat{k})$$

11.4 Angle between Two Lines

Let L_1 and L_2 be two lines passing through the origin and with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2, respectively. Let P be a point on L_1 and Q be a point on L_2. Consider the directed lines OP and OQ as given in Fig 11.6. Let θ be the acute angle between OP and OQ. Now recall that the directed line segments OP and OQ are vectors with components a_1, b_1, c_1 and a_2, b_2, c_2, respectively. Therefore, the angle θ between them is given by

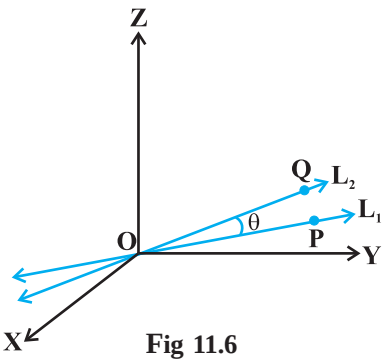

Fig 11.6

$$\cos\theta = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right| \quad \text{... (1)}$$

The angle between the lines in terms of sin θ is given by

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

$$= \sqrt{1-\frac{(a_1a_2+b_1b_2+c_1c_2)^2}{(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)}}$$

$$= \frac{\sqrt{(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)-(a_1a_2+b_1b_2+c_1c_2)^2}}{\sqrt{(a_1^2+b_1^2+c_1^2)}\sqrt{(a_2^2+b_2^2+c_2^2)}}$$

$$= \frac{\sqrt{(a_1b_2-a_2b_1)^2+(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2}}{\sqrt{a_1^2+b_1^2+c_1^2}\ \sqrt{a_2^2+b_2^2+c_2^2}} \quad \text{... (2)}$$

Note In case the lines L_1 and L_2 do not pass through the origin, we may take lines L_1' and L_2' which are parallel to L_1 and L_2 respectively and pass through the origin.

If instead of direction ratios for the lines L_1 and L_2, direction cosines, namely, l_1, m_1, n_1 for L_1 and l_2, m_2, n_2 for L_2 are given, then (1) and (2) takes the following form:

$$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2| \quad (\text{as } l_1^2 + m_1^2 + n_1^2 = 1 = l_2^2 + m_2^2 + n_2^2) \qquad \ldots (3)$$

and
$$\sin\theta = \sqrt{(l_1 m_2 - l_2 m_1)^2 - (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2} \qquad \ldots (4)$$

Two lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 are

(i) perpendicular i.e. if $\theta = 90°$ by (1)

$$\boldsymbol{a_1 a_2 + b_1 b_2 + c_1 c_2 = 0}$$

(ii) parallel i.e. if $\theta = 0$ by (2)

$$\frac{\boldsymbol{a_1}}{\boldsymbol{a_2}} = \frac{\boldsymbol{b_1}}{\boldsymbol{b_2}} = \frac{\boldsymbol{c_1}}{\boldsymbol{c_2}}$$

Now, we find the angle between two lines when their equations are given. If θ is acute the angle between the lines

$$\vec{r} = \vec{a}_1 + \lambda \vec{b}_1 \text{ and } \vec{r} = \vec{a}_2 + \mu \vec{b}_2$$

then
$$\cos\theta = \left| \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|} \right|$$

In Cartesian form, if θ is the angle between the lines

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} \qquad \ldots (1)$$

and
$$\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2} \qquad \ldots (2)$$

where, a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of the lines (1) and (2), respectively, then

$$\cos\theta = \left| \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

Example 9 Find the angle between the pair of lines given by

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$$

and
$$\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$$

Solution Here $\vec{b}_1 = \hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b}_2 = 3\hat{i} + 2\hat{j} + 6\hat{k}$

The angle θ between the two lines is given by

$$\cos\theta = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right| = \left|\frac{(\hat{i}+2\hat{j}+2\hat{k})\cdot(3\hat{i}+2\hat{j}+6\hat{k})}{\sqrt{1+4+4}\,\sqrt{9+4+36}}\right|$$

$$= \left|\frac{3+4+12}{3\times 7}\right| = \frac{19}{21}$$

Hence $\theta = \cos^{-1}\left(\frac{19}{21}\right)$

Example 10 Find the angle between the pair of lines

$$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}$$

and

$$\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$$

Solution The direction ratios of the first line are 3, 5, 4 and the direction ratios of the second line are 1, 1, 2. If θ is the angle between them, then

$$\cos\theta = \left|\frac{3.1+5.1+4.2}{\sqrt{3^2+5^2+4^2}\,\sqrt{1^2+1^2+2^2}}\right| = \frac{16}{\sqrt{50}\,\sqrt{6}} = \frac{16}{5\sqrt{2}\,\sqrt{6}} = \frac{8\sqrt{3}}{15}$$

Hence, the required angle is $\cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$.

11.5 Shortest Distance between Two Lines

If two lines in space intersect at a point, then the shortest distance between them is zero. Also, if two lines in space are parallel, then the shortest distance between them will be the perpendicular distance, i.e. the length of the perpendicular drawn from a point on one line onto the other line.

Further, in a space, there are lines which are neither intersecting nor parallel. In fact, such pair of lines are *non coplanar* and are called *skew lines*. For example, let us consider a room of size 1, 3, 2 units along *x*, *y* and *z*-axes respectively Fig 11.7.

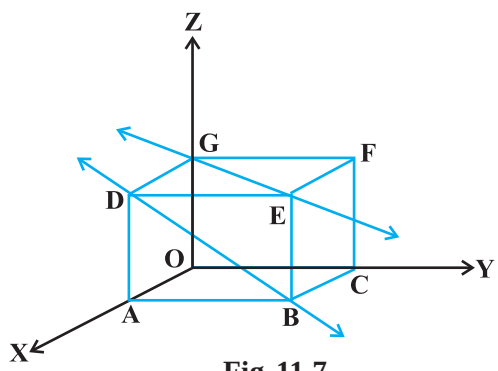

Fig 11.7

The line GE that goes diagonally across the ceiling and the line DB passes through one corner of the ceiling directly above A and goes diagonally down the wall. These lines are skew because they are not parallel and also never meet.

By the shortest distance between two lines we mean the join of a point in one line with one point on the other line so that the length of the segment so obtained is the smallest.

For skew lines, the line of the shortest distance will be perpendicular to both the lines.

11.5.1 *Distance between two skew lines*

We now determine the shortest distance between two skew lines in the following way: Let l_1 and l_2 be two skew lines with equations (Fig. 11.8)

$$\vec{r} = \vec{a}_1 + \lambda \vec{b}_1 \qquad \text{... (1)}$$

and

$$\vec{r} = \vec{a}_2 + \mu \vec{b}_2 \qquad \text{... (2)}$$

Take any point S on l_1 with position vector $\vec{a}_1$ and T on l_2, with position vector $\vec{a}_2$. Then the magnitude of the shortest distance vector will be equal to that of the projection of ST along the direction of the line of shortest distance (See 10.6.2).

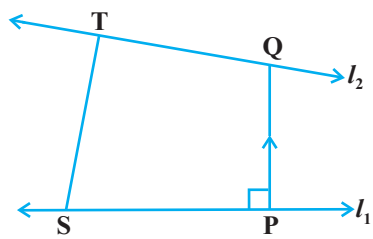

Fig 11.8

If $\overrightarrow{PQ}$ is the shortest distance vector between l_1 and l_2, then it being perpendicular to both $\vec{b}_1$ and $\vec{b}_2$, the unit vector $\hat{n}$ along $\overrightarrow{PQ}$ would therefore be

$$\hat{n} = \frac{\vec{b}_1 \times \vec{b}_2}{|\vec{b}_1 \times \vec{b}_2|} \qquad \text{... (3)}$$

Then

$$\overrightarrow{PQ} = d\,\hat{n}$$

where, d is the magnitude of the shortest distance vector. Let θ be the angle between $\overrightarrow{ST}$ and $\overrightarrow{PQ}$. Then

$$PQ = ST\,|\cos\theta|$$

But

$$\cos\theta = \left|\frac{\overrightarrow{PQ}\cdot\overrightarrow{ST}}{|\overrightarrow{PQ}|\,|\overrightarrow{ST}|}\right|$$

$$= \left|\frac{d\,\hat{n}\cdot(\vec{a}_2 - \vec{a}_1)}{d\,ST}\right| \qquad \text{(since } \overrightarrow{ST} = \vec{a}_2 - \vec{a}_1\text{)}$$

$$= \left|\frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{ST\,|\vec{b}_1 \times \vec{b}_2|}\right| \qquad \text{[From (3)]}$$

Hence, the required shortest distance is

$$d = \text{PQ} = \text{ST}\,|\cos\theta|$$

or

$$\boldsymbol{d} = \left| \frac{(\vec{b}_1 \times \vec{b}_2)\,.\,(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

Cartesian form

The shortest distance between the lines

$$l_1 : \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

and

$$l_2 : \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

is

$$\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

11.5.2 *Distance between parallel lines*

If two lines l_1 and l_2 are parallel, then they are coplanar. Let the lines be given by

$$\vec{r} = \vec{a}_1 + \lambda\,\vec{b} \qquad \text{... (1)}$$

and

$$\vec{r} = \vec{a}_2 + \mu\,\vec{b} \qquad \text{... (2)}$$

where, $\vec{a}_1$ is the position vector of a point S on l_1 and $\vec{a}_2$ is the position vector of a point T on l_2 Fig 11.9.

As l_1, l_2 are coplanar, if the foot of the perpendicular from T on the line l_1 is P, then the distance between the lines l_1 and l_2 = |TP|.

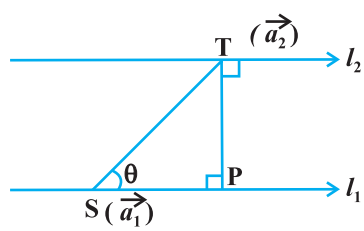

Fig 11.9

Let θ be the angle between the vectors $\overrightarrow{\text{ST}}$ and $\vec{b}$. Then

$$\vec{b} \times \overrightarrow{\text{ST}} = (|\vec{b}|\,|\overrightarrow{\text{ST}}|\sin\theta)\,\hat{n} \qquad \text{... (3)}$$

where $\hat{n}$ is the unit vector perpendicular to the plane of the lines l_1 and l_2.

But

$$\overrightarrow{\text{ST}} = \vec{a}_2 - \vec{a}_1$$

Therefore, from (3), we get

$$\vec{b} \times (\vec{a}_2 - \vec{a}_1) = |\vec{b}|\,\text{PT}\,\hat{n} \qquad \text{(since PT = ST sin } \theta\text{)}$$

i.e.,

$$|\vec{b} \times (\vec{a}_2 - \vec{a}_1)| = |\vec{b}|\,\text{PT} \cdot 1 \qquad \text{(as } |\hat{n}| = 1\text{)}$$

Hence, the distance between the given parallel lines is

$$\boldsymbol{d = |\overrightarrow{\text{PT}}| = \left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right|}$$

Example 11 Find the shortest distance between the lines l_1 and l_2 whose vector equations are

$$\vec{r} = \hat{i} + \hat{j} + \lambda\,(2\hat{i} - \hat{j} + \hat{k}) \qquad \text{... (1)}$$

and

$$\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu\,(3\hat{i} - 5\hat{j} + 2\hat{k}) \qquad \text{... (2)}$$

Solution Comparing (1) and (2) with $\vec{r} = \vec{a}_1 + \lambda\,\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\,\vec{b}_2$ respectively,

we get

$$\vec{a}_1 = \hat{i} + \hat{j}, \quad \vec{b}_1 = 2\hat{i} - \hat{j} + \hat{k}$$

$$\vec{a}_2 = 2\hat{i} + \hat{j} - \hat{k} \text{ and } \vec{b}_2 = 3\hat{i} - 5\hat{j} + 2\hat{k}$$

Therefore

$$\vec{a}_2 - \vec{a}_1 = \hat{i} - \hat{k}$$

and

$$\vec{b}_1 \times \vec{b}_2 = (2\hat{i} - \hat{j} + \hat{k}) \times (3\hat{i} - 5\hat{j} + 2\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix} = 3\hat{i} - \hat{j} - 7\hat{k}$$

So

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{9 + 1 + 49} = \sqrt{59}$$

Hence, the shortest distance between the given lines is given by

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right| = \frac{|3 - 0 + 7|}{\sqrt{59}} = \frac{10}{\sqrt{59}}$$

Example 12 Find the distance between the lines l_1 and l_2 given by

$$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda\,(2\hat{i} + 3\hat{j} + 6\hat{k})$$

and

$$\vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu\,(2\hat{i} + 3\hat{j} + 6\hat{k})$$

Solution The two lines are parallel (Why?) We have

$$\vec{a}_1 = \hat{i} + 2\hat{j} - 4\hat{k},\ \vec{a}_2 = 3\hat{i} + 3\hat{j} - 5\hat{k} \text{ and } \vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

Therefore, the distance between the lines is given by

$$d = \left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right| = \left| \frac{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix}}{\sqrt{4+9+36}} \right|$$

or $$= \frac{|-9\hat{i} + 14\hat{j} - 4\hat{k}|}{\sqrt{49}} = \frac{\sqrt{293}}{\sqrt{49}} = \frac{\sqrt{293}}{7}$$

EXERCISE 11.2

1. Show that the three lines with direction cosines

   $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \quad \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \quad \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ are mutually perpendicular.

2. Show that the line through the points (1, – 1, 2), (3, 4, – 2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).

3. Show that the line through the points (4, 7, 8), (2, 3, 4) is parallel to the line through the points (– 1, – 2, 1), (1, 2, 5).

4. Find the equation of the line which passes through the point (1, 2, 3) and is parallel to the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$.

5. Find the equation of the line in vector and in cartesian form that passes through the point with position vector $2\hat{i} - j + 4\hat{k}$ and is in the direction $\hat{i} + 2\hat{j} - \hat{k}$.

6. Find the cartesian equation of the line which passes through the point (– 2, 4, – 5) and parallel to the line given by $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$.

7. The cartesian equation of a line is $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$. Write its vector form.

8. Find the vector and the cartesian equations of the lines that passes through the origin and (5, – 2, 3).

9. Find the vector and the cartesian equations of the line that passes through the points (3, – 2, – 5), (3, – 2, 6).

10. Find the angle between the following pairs of lines:

(i) $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$ and

$\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$

(ii) $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} - 2\hat{k})$ and

$\vec{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$

11. Find the angle between the following pair of lines:

(i) $\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}$ and $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$

(ii) $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and $\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$

12. Find the values of p so that the lines $\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$

and $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are at right angles.

13. Show that the lines $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ are perpendicular to each other.

14. Find the shortest distance between the lines

$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and

$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$

15. Find the shortest distance between the lines

$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

16. Find the shortest distance between the lines whose vector equations are

$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$

and $\vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$

17. Find the shortest distance between the lines whose vector equations are

$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$ and

$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$

11.6 Plane

A plane is determined uniquely if any one of the following is known:

(i) the normal to the plane and its distance from the origin is given, i.e., equation of a plane in normal form.

(ii) it passes through a point and is perpendicular to a given direction.

(iii) it passes through three given non collinear points.

Now we shall find vector and Cartesian equations of the planes.

11.6.1 *Equation of a plane in normal form*

Consider a plane whose perpendicular distance from the origin is d ($d \neq 0$). Fig 11.10.

If $\overrightarrow{ON}$ is the normal from the origin to the plane, and $\hat{n}$ is the unit normal vector along $\overrightarrow{ON}$. Then $\overrightarrow{ON} = d\,\hat{n}$. Let P be any point on the plane. Therefore, $\overrightarrow{NP}$ is perpendicular to $\overrightarrow{ON}$.

Therefore, $\overrightarrow{NP} \cdot \overrightarrow{ON} = 0$ ... (1)

Let $\vec{r}$ be the position vector of the point P, then $\overrightarrow{NP} = \vec{r} - d\,\hat{n}$ (as $\overrightarrow{ON} + \overrightarrow{NP} = \overrightarrow{OP}$)

Therefore, (1) becomes

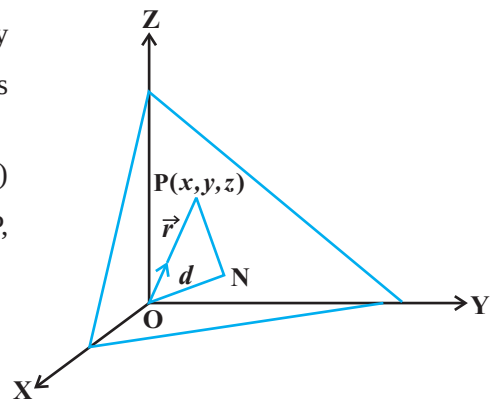

Fig 11.10

$$(\vec{r} - d\,\hat{n}) \cdot d\,\hat{n} = 0$$

or $$(\vec{r} - d\,\hat{n}) \cdot \hat{n} = 0 \qquad (d \neq 0)$$

or $$\vec{r} \cdot \hat{n} - d\,\hat{n} \cdot \hat{n} = 0$$

i.e., $$\boldsymbol{\vec{r} \cdot \hat{n} = d} \qquad (\text{as } \hat{n} \cdot \hat{n} = 1) \qquad \ldots (2)$$

This is the vector form of the equation of the plane.

Cartesian form

Equation (2) gives the vector equation of a plane, where $\hat{n}$ is the unit vector normal to the plane. Let P(x, y, z) be any point on the plane. Then

$$\overrightarrow{OP} = \vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

Let l, m, n be the direction cosines of $\hat{n}$. Then

$$\hat{n} = l\,\hat{i} + m\,\hat{j} + n\,\hat{k}$$

Therefore, (2) gives

$$(x\,\hat{i} + y\,\hat{j} + z\,\hat{k})\cdot(l\,\hat{i} + m\,\hat{j} + n\,\hat{k}) = d$$

i.e., $$\boldsymbol{lx + my + nz = d} \qquad \text{... (3)}$$

This is the cartesian equation of the plane in the normal form.

Note Equation (3) shows that if $\vec{r}\cdot(a\,\hat{i} + b\,\hat{j} + c\,\hat{k}) = d$ is the vector equation of a plane, then $ax + by + cz = d$ is the Cartesian equation of the plane, where a, b and c are the direction ratios of the normal to the plane.

Example 13 Find the vector equation of the plane which is at a distance of $\frac{6}{\sqrt{29}}$ from the origin and its normal vector from the origin is $2\hat{i} - 3\hat{j} + 4\hat{k}$. Also find its cartesian form.

Solution Let $\vec{n} = 2\,\hat{i} - 3\,\hat{j} + 4\,\hat{k}$. Then

$$\hat{n} = \frac{\vec{n}}{|\,\vec{n}\,|} = \frac{2\,\hat{i} - 3\,\hat{j} + 4\,\hat{k}}{\sqrt{4+9+16}} = \frac{2\,\hat{i} - 3\,\hat{j} + 4\,\hat{k}}{\sqrt{29}}$$

Hence, the required equation of the plane is

$$\vec{r}\cdot\left(\frac{2}{\sqrt{29}}\,\hat{i} + \frac{-3}{\sqrt{29}}\,\hat{j} + \frac{4}{\sqrt{29}}\,\hat{k}\right) = \frac{6}{\sqrt{29}}$$

Example 14 Find the direction cosines of the unit vector perpendicular to the plane

$\vec{r}\cdot(6\,\hat{i} - 3\,\hat{j} - 2\,\hat{k}) + 1 = 0$ passing through the origin.

Solution The given equation can be written as

$$\vec{r}\cdot(-6\,\hat{i} + 3\,\hat{j} + 2\,\hat{k}) = 1 \qquad \text{... (1)}$$

Now $$|-6\,\hat{i} + 3\,\hat{j} + 2\,\hat{k}\,| = \sqrt{36+9+4} = 7$$

Therefore, dividing both sides of (1) by 7, we get

$$\vec{r}\cdot\left(-\frac{6}{7}\,\hat{i} + \frac{3}{7}\,\hat{j} + \frac{2}{7}\,\hat{k}\right) = \frac{1}{7}$$

which is the equation of the plane in the form $\vec{r}\cdot\hat{n} = d$.

This shows that $\hat{n} = -\frac{6}{7}\,\hat{i} + \frac{3}{7}\,\hat{j} + \frac{2}{7}\,\hat{k}$ is a unit vector perpendicular to the plane through the origin. Hence, the direction cosines of $\hat{n}$ are $\frac{-6}{7}, \frac{3}{7}, \frac{2}{7}$.

Example 15 Find the distance of the plane $2x - 3y + 4z - 6 = 0$ from the origin.

Solution Since the direction ratios of the normal to the plane are 2, –3, 4; the direction cosines of it are

$$\frac{2}{\sqrt{2^2+(-3)^2+4^2}}, \frac{-3}{\sqrt{2^2+(-3)^2+4^2}}, \frac{4}{\sqrt{2^2+(-3)^2+4^2}} \text{, i.e., } \frac{2}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{4}{\sqrt{29}}$$

Hence, dividing the equation $2x - 3y + 4z - 6 = 0$ i.e., $2x - 3y + 4z = 6$ throughout by $\sqrt{29}$, we get

$$\frac{2}{\sqrt{29}}x + \frac{-3}{\sqrt{29}}y + \frac{4}{\sqrt{29}}z = \frac{6}{\sqrt{29}}$$

This is of the form $lx + my + nz = d$, where d is the distance of the plane from the origin. So, the distance of the plane from the origin is $\frac{6}{\sqrt{29}}$.

Example 16 Find the coordinates of the foot of the perpendicular drawn from the origin to the plane $2x - 3y + 4z - 6 = 0$.

Solution Let the coordinates of the foot of the perpendicular P from the origin to the plane is (x_1, y_1, z_1) (Fig 11.11).

Then, the direction ratios of the line OP are x_1, y_1, z_1.

Writing the equation of the plane in the normal form, we have

$$\frac{2}{\sqrt{29}}x - \frac{3}{\sqrt{29}}y + \frac{4}{\sqrt{29}}z = \frac{6}{\sqrt{29}}$$

where, $\frac{2}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{4}{\sqrt{29}}$ are the direction cosines of the OP.

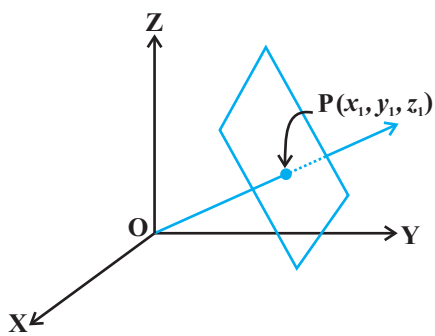

Fig 11.11

Since *d.c.'s* and direction ratios of a line are proportional, we have

$$\frac{x_1}{\frac{2}{\sqrt{29}}} = \frac{y_1}{\frac{-3}{\sqrt{29}}} = \frac{z_1}{\frac{4}{\sqrt{29}}} = k$$

i.e.,

$$x_1 = \frac{2k}{\sqrt{29}}, y_1 = \frac{-3k}{\sqrt{29}}, z_1 = \frac{4k}{\sqrt{29}}$$

Substituting these in the equation of the plane, we get $k = \frac{6}{\sqrt{29}}$.

Hence, the foot of the perpendicular is $\left(\frac{12}{29}, \frac{-18}{29}, \frac{24}{29}\right)$.

Note If d is the distance from the origin and l, m, n are the direction cosines of the normal to the plane through the origin, then the foot of the perpendicular is (ld, md, nd).

11.6.2 *Equation of a plane perpendicular to a given vector and passing through a given point*

In the space, there can be many planes that are perpendicular to the given vector, but through a given point $P(x_1, y_1, z_1)$, only one such plane exists (see Fig 11.12).

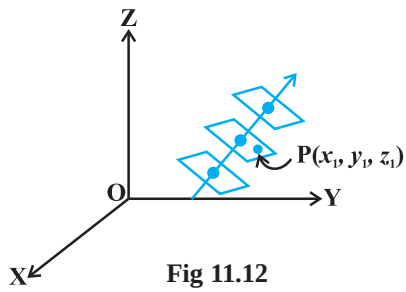

Fig 11.12

Let a plane pass through a point A with position vector $\vec{a}$ and perpendicular to the vector $\vec{N}$.

Let $\vec{r}$ be the position vector of any point $P(x, y, z)$ in the plane. (Fig 11.13).

Then the point P lies in the plane if and only if $\overrightarrow{AP}$ is perpendicular to $\vec{N}$. i.e., $\overrightarrow{AP} \cdot \vec{N} = 0$. But $\overrightarrow{AP} = \vec{r} - \vec{a}$. Therefore, $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$ … (1)

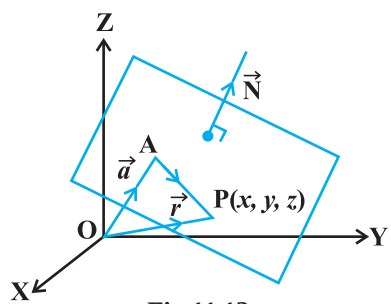

Fig 11.13

This is the vector equation of the plane.

Cartesian form

Let the given point A be (x_1, y_1, z_1), P be (x, y, z) and direction ratios of $\vec{N}$ are A, B and C. Then,

$$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}, \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \text{ and } \vec{N} = A\hat{i} + B\hat{j} + C\hat{k}$$

Now $\quad (\vec{r} - \vec{a}) \cdot \vec{N} = 0$

So $\quad \left[(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}\right] \cdot (A\hat{i} + B\hat{j} + C\hat{k}) = 0$

i.e. $\quad \mathbf{A(x - x_1) + B(y - y_1) + C(z - z_1) = 0}$

Example 17 Find the vector and cartesian equations of the plane which passes through the point (5, 2, – 4) and perpendicular to the line with direction ratios 2, 3, – 1.

Solution We have the position vector of point (5, 2, – 4) as $\vec{a} = 5\hat{i} + 2\hat{j} - 4\hat{k}$ and the normal vector $\vec{N}$ perpendicular to the plane as $\vec{N} = 2\hat{i} + 3\hat{j} - \hat{k}$

Therefore, the vector equation of the plane is given by $(\vec{r} - \vec{a}).\vec{N} = 0$

or $$[\vec{r} - (5\hat{i} + 2\hat{j} - 4\hat{k})] \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = 0 \quad \text{... (1)}$$

Transforming (1) into Cartesian form, we have

$$[(x-5)\hat{i} + (y-2)\hat{j} + (z+4)\hat{k}] \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = 0$$

or $$2(x-5) + 3(y-2) - 1(z+4) = 0$$

i.e. $$2x + 3y - z = 20$$

which is the cartesian equation of the plane.

11.6.3 *Equation of a plane passing through three non collinear points*

Let R, S and T be three non collinear points on the plane with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively (Fig 11.14).

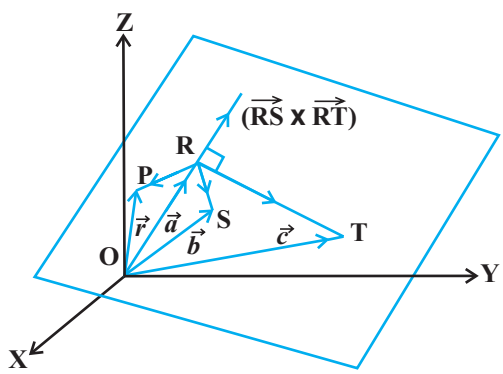

Fig 11.14

The vectors $\overrightarrow{RS}$ and $\overrightarrow{RT}$ are in the given plane. Therefore, the vector $\overrightarrow{RS} \times \overrightarrow{RT}$ is perpendicular to the plane containing points R, S and T. Let $\vec{r}$ be the position vector of any point P in the plane. Therefore, the equation of the plane passing through R and perpendicular to the vector $\overrightarrow{RS} \times \overrightarrow{RT}$ is

$$(\vec{r} - \vec{a}) \cdot (\overrightarrow{RS} \times \overrightarrow{RT}) = 0$$

or $$\mathbf{(\vec{r} - \vec{a}) \times [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0} \quad \text{... (1)}$$

This is the equation of the plane in vector form passing through three noncollinear points.

Note Why was it necessary to say that the three points had to be non collinear? If the three points were on the same line, then there will be many planes that will contain them (Fig 11.15).

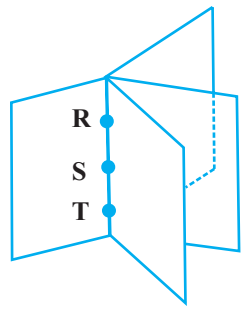

Fig 11.15

These planes will resemble the pages of a book where the line containing the points R, S and T are members in the binding of the book.

Cartesian form

Let (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) be the coordinates of the points R, S and T respectively. Let (x, y, z) be the coordinates of any point P on the plane with position vector $\vec{r}$. Then

$$\overrightarrow{RP} = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$$

$$\overrightarrow{RS} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\overrightarrow{RT} = (x_3 - x_1)\hat{i} + (y_3 - y_1)\hat{j} + (z_3 - z_1)\hat{k}$$

Substituting these values in equation (1) of the vector form and expressing it in the form of a determinant, we have

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

which is the equation of the plane in Cartesian form passing through three non collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

Example 18 Find the vector equations of the plane passing through the points R(2, 5, – 3), S(– 2, – 3, 5) and T(5, 3,– 3).

Solution Let $\vec{a} = 2\hat{i} + 5\hat{j} - 3\hat{k}$, $\vec{b} = -2\hat{i} - 3\hat{j} + 5\hat{k}$, $\vec{c} = 5\hat{i} + 3\hat{j} - 3\hat{k}$

Then the vector equation of the plane passing through $\vec{a}$, $\vec{b}$ and $\vec{c}$ and is given by

$$(\vec{r} - \vec{a}) \cdot (\overrightarrow{RS} \times \overrightarrow{RT}) = 0 \quad \text{(Why?)}$$

or $$(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$$

i.e. $$[\vec{r} - (2\hat{i} + 5\hat{j} - 3\hat{k})] \cdot [(-4\hat{i} - 8\hat{j} + 8\hat{k}) \times (3\hat{i} - 2\hat{j})] = 0$$

11.6.4 *Intercept form of the equation of a plane*

In this section, we shall deduce the equation of a plane in terms of the intercepts made by the plane on the coordinate axes. Let the equation of the plane be

$$Ax + By + Cz + D = 0 \quad (D \neq 0) \qquad \text{... (1)}$$

Let the plane make intercepts a, b, c on x, y and z axes, respectively (Fig 11.16).

Hence, the plane meets x, y and z-axes at $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, respectively.

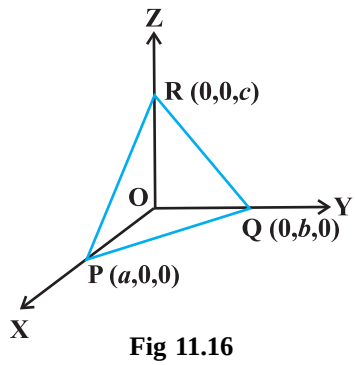

Fig 11.16

Therefore
$$Aa + D = 0 \text{ or } A = \frac{-D}{a}$$
$$Bb + D = 0 \text{ or } B = \frac{-D}{b}$$
$$Cc + D = 0 \text{ or } C = \frac{-D}{c}$$

Substituting these values in the equation (1) of the plane and simplifying, we get

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \text{... (1)}$$

which is the required equation of the plane in the intercept form.

Example 19 Find the equation of the plane with intercepts 2, 3 and 4 on the x, y and z-axis respectively.

Solution Let the equation of the plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \text{... (1)}$$

Here $a = 2, b = 3, c = 4.$

Substituting the values of a, b and c in (1), we get the required equation of the plane as $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$ or $6x + 4y + 3z = 12$.

11.6.5 *Plane passing through the intersection of two given planes*

Let π_1 and π_2 be two planes with equations $\vec{r} \cdot \hat{n}_1 = d_1$ and $\vec{r} \cdot \hat{n}_2 = d_2$ respectively. The position vector of any point on the line of intersection must satisfy both the equations (Fig 11.17).

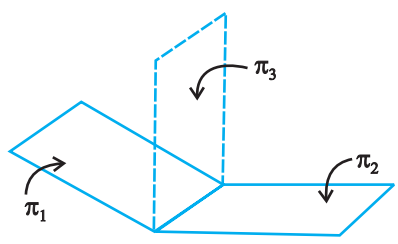

Fig 11.17

If $\vec{t}$ is the position vector of a point on the line, then

$$\vec{t}\cdot\hat{n}_1 = d_1 \text{ and } \vec{t}\cdot\hat{n}_2 = d_2$$

Therefore, for all real values of λ, we have

$$\vec{t}\cdot(\hat{n}_1+\lambda\hat{n}_2) = d_1+\lambda d_2$$

Since $\vec{t}$ is arbitrary, it satisfies for any point on the line.

Hence, the equation $\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2)=d_1+\lambda d_2$ represents a plane π_3 which is such that if any vector $\vec{r}$ satisfies both the equations π_1 and π_2, it also satisfies the equation π_3 i.e., any plane passing through the intersection of the planes

$$\vec{r}\cdot\vec{n}_1 = d_1 \text{ and } \vec{r}\cdot\vec{n}_2 = d_2$$

has the equation $$\boldsymbol{\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2) = d_1 + \lambda d_2} \quad \text{... (1)}$$

Cartesian form

In Cartesian system, let

$$\vec{n}_1 = A_1\hat{i}+B_2\hat{j}+C_1\hat{k}$$

$$\vec{n}_2 = A_2\hat{i}+B_2\hat{j}+C_2\hat{k}$$

and $$\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$$

Then (1) becomes

$$x(A_1+\lambda A_2)+y(B_1+\lambda B_2)+z(C_1+\lambda C_2) = d_1+\lambda d_2$$

or $$\boldsymbol{(A_1x+B_1y+C_1z-d_1)+\lambda(A_2x+B_2y+C_2z-d_2)=0} \quad \text{... (2)}$$

which is the required Cartesian form of the equation of the plane passing through the intersection of the given planes for each value of λ.

Example 20 Find the vector equation of the plane passing through the intersection of the planes $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=6$ and $\vec{r}\cdot(2\hat{i}+3\hat{j}+4\hat{k})=-5$, and the point (1, 1, 1).

Solution Here, $\vec{n}_1=\hat{i}+\hat{j}+\hat{k}$ and $\vec{n}_2 = 2\hat{i}+3\hat{j}+4\hat{k}$;

and $d_1 = 6$ and $d_2 = -5$

Hence, using the relation $\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2)=d_1+\lambda d_2$, we get

$$\vec{r}\cdot[\hat{i}+\hat{j}+\hat{k}+\lambda(2\hat{i}+3\hat{j}+4\hat{k})] = 6-5\lambda$$

or $$\vec{r}\cdot[(1+2\lambda)\hat{i}+(1+3\lambda)\hat{j}+(1+4\lambda)\hat{k}] = 6-5\lambda \quad \text{... (1)}$$

where, λ is some real number.

Taking $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, we get

$$(x\hat{i} + y\hat{j} + z\hat{k})\cdot[(1+2\lambda)\hat{i} + (1+3\lambda)\hat{j} + (1+4\lambda)\hat{k}] = 6 - 5\lambda$$

or $(1 + 2\lambda)\, x + (1 + 3\lambda)\, y + (1 + 4\lambda)\, z = 6 - 5\lambda$

or $(x + y + z - 6) + \lambda\ (2x + 3y + 4z + 5) = 0$ ... (2)

Given that the plane passes through the point (1,1,1), it must satisfy (2), i.e.

$$(1 + 1 + 1 - 6) + \lambda\,(2 + 3 + 4 + 5) = 0$$

or $\lambda = \dfrac{3}{14}$

Putting the values of λ in (1), we get

$$\vec{r}\left[\left(1+\frac{3}{7}\right)\hat{i} + \left(1+\frac{9}{14}\right)\hat{j} + \left(1+\frac{6}{7}\right)\hat{k}\right] = 6 - \frac{15}{14}$$

or $\vec{r}\left(\dfrac{10}{7}\hat{i} + \dfrac{23}{14}\hat{j} + \dfrac{13}{7}\hat{k}\right) = \dfrac{69}{14}$

or $\vec{r}\cdot(20\hat{i} + 23\hat{j} + 26\hat{k}) = 69$

which is the required vector equation of the plane.

11.7 Coplanarity of Two Lines

Let the given lines be

$$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1 \quad \text{... (1)}$$

and $$\vec{r} = \vec{a}_2 + \mu\vec{b}_2 \quad \text{... (2)}$$

The line (1) passes through the point, say A, with position vector $\vec{a}_1$ and is parallel to $\vec{b}_1$. The line (2) passes through the point, say B with position vector $\vec{a}_2$ and is parallel to $\vec{b}_2$.

Thus, $\overrightarrow{AB} = \vec{a}_2 - \vec{a}_1$

The given lines are coplanar if and only if $\overrightarrow{AB}$ is perpendicular to $\vec{b}_1 \times \vec{b}_2$.

i.e. $\overrightarrow{AB}.(\vec{b}_1 \times \vec{b}_2) = 0$ or $(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2) = 0$

Cartesian form

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the coordinates of the points A and B respectively.

Let a_1, b_1, c_1 and a_2, b_2, c_2 be the direction ratios of $\vec{b}_1$ and $\vec{b}_2$, respectively. Then

$$\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\vec{b}_1 = a_1\hat{i} + b_1\hat{j} + c_1\hat{k} \text{ and } \vec{b}_2 = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$$

The given lines are coplanar if and only if $\overrightarrow{AB} \cdot (\vec{b}_1 \times \vec{b}_2) = 0$. In the cartesian form, it can be expressed as

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_2 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0 \qquad \text{... (4)}$$

Example 21 Show that the lines

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5} \text{ and } \frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5} \text{ are coplanar.}$$

Solution Here, $x_1 = -3$, $y_1 = 1$, $z_1 = 5$, $a_1 = -3$, $b_1 = 1$, $c_1 = 5$

$x_2 = -1$, $y_2 = 2$, $z_2 = 5$, $a_2 = -1$, $b_2 = 2$, $c_2 = 5$

Now, consider the determinant

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = 0$$

Therefore, lines are coplanar.

11.8 Angle between Two Planes

Definition 2 The angle between two planes is defined as the angle between their normals (Fig 11.18 (a)). Observe that if θ is an angle between the two planes, then so is 180 – θ (Fig 11.18 (b)). We shall take the acute angle as the angles between two planes.

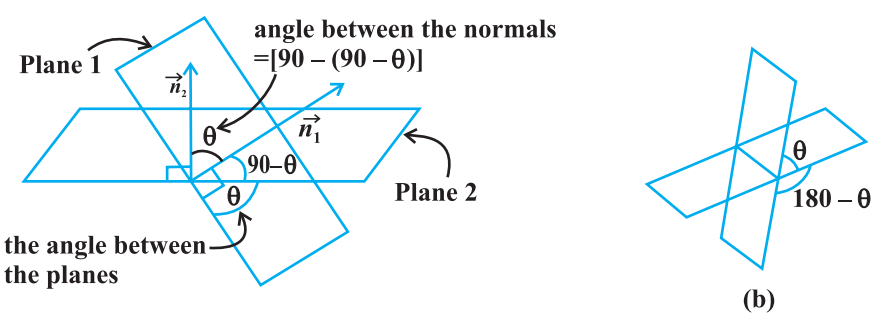

Fig 11.18

If $\vec{n}_1$ and $\vec{n}_2$ are normals to the planes and θ be the angle between the planes

$$\vec{r} \cdot \vec{n}_1 = d_1 \text{ and } \vec{r} \cdot \vec{n}_2 = d_2 .$$

Then θ is the angle between the normals to the planes drawn from some common point.

We have, $$\cos\theta = \left| \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} \right|$$

Note The planes are perpendicular to each other if $\vec{n}_1 . \vec{n}_2 = 0$ and parallel if $\vec{n}_1$ is parallel to $\vec{n}_2$.

Cartesian form Let θ be the angle between the planes,

$$A_1 x + B_1 y + C_1 z + D_1 = 0 \text{ and } A_2 x + B_2 y + C_2 z + D_2 = 0$$

The direction ratios of the normal to the planes are A_1, B_1, C_1 and A_2, B_2, C_2 respectively.

Therefore, $$\cos\theta = \left| \frac{A_1 A_2 + B_1 B_2 + C_1 C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\ \sqrt{A_2^2 + B_2^2 + C_2^2}} \right|$$

Note

1. If the planes are at right angles, then θ = 90° and so cos θ = 0. Hence, $\cos\theta = A_1A_2 + B_1B_2 + C_1C_2 = 0$.
2. If the planes are parallel, then $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$.

Example 22 Find the angle between the two planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$ using vector method.

Solution The angle between two planes is the angle between their normals. From the equation of the planes, the normal vectors are

$$\vec{N}_1 = 2\hat{i} + \hat{j} - 2\hat{k} \text{ and } \vec{N}_2 = 3\hat{i} - 6\hat{j} - 2\hat{k}$$

Therefore $$\cos\theta = \left| \frac{\vec{N}_1 \cdot \vec{N}_2}{|\vec{N}_1|\,|\vec{N}_2|} \right| = \left| \frac{(2\hat{i} + \hat{j} - 2\hat{k}) \cdot (3\hat{i} - 6\hat{j} - 2\hat{k})}{\sqrt{4+1+4}\ \sqrt{9+36+4}} \right| = \left(\frac{4}{21}\right)$$

Hence $$\theta = \cos^{-1}\left(\frac{4}{21}\right)$$

Example 23 Find the angle between the two planes $3x - 6y + 2z = 7$ and $2x + 2y - 2z = 5$.

Solution Comparing the given equations of the planes with the equations

$$A_1 x + B_1 y + C_1 z + D_1 = 0 \text{ and } A_2 x + B_2 y + C_2 z + D_2 = 0$$

We get
$$A_1 = 3, B_1 = -6, C_1 = 2$$
$$A_2 = 2, B_2 = 2, C_2 = -2$$

$$\cos\theta = \left| \frac{3\times 2 + (-6)(2) + (2)(-2)}{\sqrt{(3^2 + (-6)^2 + (-2)^2)}\sqrt{(2^2 + 2^2 + (-2)^2)}} \right|$$

$$= \left| \frac{-10}{7\times 2\sqrt{3}} \right| = \frac{5}{7\sqrt{3}} = \frac{5\sqrt{3}}{21}$$

Therefore, $\theta = \cos^{-1}\left(\frac{5\sqrt{3}}{21}\right)$

11.9 Distance of a Point from a Plane

Vector form

Consider a point P with position vector $\vec{a}$ and a plane π_1 whose equation is $\vec{r}\cdot\hat{n} = d$ (Fig 11.19).

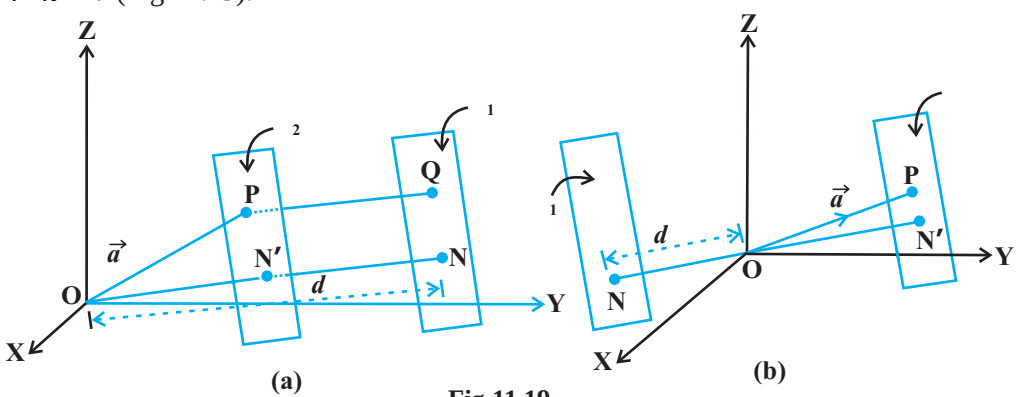

Fig 11.19

Consider a plane π_2 through P parallel to the plane π_1. The unit vector normal to π_2 is $\hat{n}$. Hence, its equation is $(\vec{r} - \vec{a})\cdot\hat{n} = 0$

i.e., $$\vec{r}\cdot\hat{n} = \vec{a}\cdot\hat{n}$$

Thus, the distance ON′ of this plane from the origin is $|\vec{a}\cdot\hat{n}|$. Therefore, the distance PQ from the plane π_1 is (Fig. 11.21 (a))

i.e., $$ON - ON' = |d - \vec{a}\cdot\hat{n}|$$

which is the length of the perpendicular from a point to the given plane.

We may establish the similar results for (Fig 11.19 (b)).

Note

1. If the equation of the plane π_2 is in the form $\vec{r} \cdot \vec{N} = d$, where $\vec{N}$ is normal to the plane, then the perpendicular distance is $\dfrac{|\vec{a} \cdot \vec{N} - d|}{|\vec{N}|}$.

2. The length of the perpendicular from origin O to the plane $\vec{r} \cdot \vec{N} = d$ is $\dfrac{|d|}{|\vec{N}|}$ (since $\vec{a} = 0$).

Cartesian form

Let $P(x_1, y_1, z_1)$ be the given point with position vector $\vec{a}$ and

$$Ax + By + Cz = D$$

be the Cartesian equation of the given plane. Then

$$\vec{a} = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}$$

$$\vec{N} = A\hat{i} + B\hat{j} + C\hat{k}$$

Hence, from Note 1, the perpendicular from P to the plane is

$$\left| \frac{(x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}) \cdot (A\hat{i} + B\hat{j} + C\hat{k}) - D}{\sqrt{A^2 + B^2 + C^2}} \right|$$

$$= \left| \frac{A x_1 + B y_1 + C z_1 - D}{\sqrt{A^2 + B^2 + C^2}} \right|$$

Example 24 Find the distance of a point (2, 5, – 3) from the plane

$$\vec{r} \cdot (6\hat{i} - 3\hat{j} + 2\hat{k}) = 4$$

Solution Here, $\vec{a} = 2\hat{i} + 5\hat{j} - 3\hat{k}$, $\vec{N} = 6\hat{i} - 3\hat{j} + 2\hat{k}$ and $d = 4$.

Therefore, the distance of the point (2, 5, – 3) from the given plane is

$$\frac{|(2\hat{i} + 5\hat{j} - 3\hat{k}) \cdot (6\hat{i} - 3\hat{j} + 2\hat{k}) - 4|}{|6\hat{i} - 3\hat{j} + 2\hat{k}|} = \frac{|12 - 15 - 6 - 4|}{\sqrt{36 + 9 + 4}} = \frac{13}{7}$$

11.10 Angle between a Line and a Plane

Definition 3 The angle between a line and a plane is the complement of the angle between the line and normal to the plane (Fig 11.20).

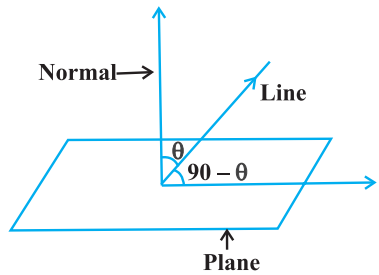

Fig 11.20

Vector form If the equation of the line is $\vec{r} = \vec{a} + \lambda \vec{b}$ and the equation of the plane is $\vec{r} \cdot \vec{n} = d$. Then the angle θ between the line and the normal to the plane is

$$\cos \theta = \left| \frac{\vec{b} \cdot \vec{n}}{|\vec{b}| \cdot |\vec{n}|} \right|$$

and so the angle ϕ between the line and the plane is given by 90 – θ, i.e.,

$$\sin (90 - \theta) = \cos \theta$$

i.e.

$$\sin \phi = \left| \frac{\vec{b} \cdot \vec{n}}{|\vec{b}| \; |\vec{n}|} \right| \text{ or } \phi = \sin^{-1} \left| \frac{\bar{b} \cdot \bar{n}}{|\bar{b}||\bar{n}|} \right|$$

Example 25 Find the angle between the line

$$\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$$

and the plane $10x + 2y - 11z = 3$.

Solution Let θ be the angle between the line and the normal to the plane. Converting the given equations into vector form, we have

$$\vec{r} = (-\hat{i} + 3\hat{k}) + \lambda (2\hat{i} + 3\hat{j} + 6\hat{k})$$

and

$$\vec{r} \cdot (10\hat{i} + 2\hat{j} - 11\hat{k}) = 3$$

Here

$$\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k} \quad \text{and} \quad \vec{n} = 10\hat{i} + 2\hat{j} - 11\hat{k}$$

$$\sin \phi = \left| \frac{(2\hat{i} + 3\hat{j} + 6\hat{k}) \cdot (10\hat{i} + 2\hat{j} - 11\hat{k})}{\sqrt{2^2 + 3^2 + 6^2} \sqrt{10^2 + 2^2 + 11^2}} \right|$$

$$= \left| \frac{-40}{7 \times 15} \right| = \left| \frac{-8}{21} \right| = \frac{8}{21} \quad \text{or } \phi = \sin^{-1}\left(\frac{8}{21}\right)$$

EXERCISE 11.3

1. In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.

(a) $z = 2$ (b) $x + y + z = 1$

(c) $2x + 3y - z = 5$ (d) $5y + 8 = 0$

2. Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector $3\hat{i} + 5\hat{j} - 6\hat{k}$.

3. Find the Cartesian equation of the following planes:

(a) $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 2$ (b) $\vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$

(c) $\vec{r} \cdot [(s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k}] = 15$

4. In the following cases, find the coordinates of the foot of the perpendicular drawn from the origin.

(a) $2x + 3y + 4z - 12 = 0$ (b) $3y + 4z - 6 = 0$

(c) $x + y + z = 1$ (d) $5y + 8 = 0$

5. Find the vector and cartesian equations of the planes

(a) that passes through the point (1, 0, – 2) and the normal to the plane is $\hat{i} + \hat{j} - \hat{k}$.

(b) that passes through the point (1,4, 6) and the normal vector to the plane is $\hat{i} - 2\hat{j} + \hat{k}$.

6. Find the equations of the planes that passes through three points.

(a) (1, 1, – 1), (6, 4, – 5), (– 4, – 2, 3)

(b) (1, 1, 0), (1, 2, 1), (– 2, 2, – 1)

7. Find the intercepts cut off by the plane $2x + y - z = 5$.

8. Find the equation of the plane with intercept 3 on the y-axis and parallel to ZOX plane.

9. Find the equation of the plane through the intersection of the planes $3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$ and the point (2, 2, 1).

10. Find the vector equation of the plane passing through the intersection of the planes $\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 7$, $\vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$ and through the point (2, 1, 3).

11. Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$.

12. Find the angle between the planes whose vector equations are

$\vec{r}\cdot(2\hat{i}+2\hat{j}-3\hat{k})=5$ and $\vec{r}\cdot(3\hat{i}-3\hat{j}+5\hat{k})=3.$

13. In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

(a) $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$

(b) $2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$

(c) $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$

(d) $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$

(e) $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$

14. In the following cases, find the distance of each of the given points from the corresponding given plane.

	Point	**Plane**
(a)	(0, 0, 0)	$3x - 4y + 12z = 3$
(b)	(3, – 2, 1)	$2x - y + 2z + 3 = 0$
(c)	(2, 3, – 5)	$x + 2y - 2z = 9$
(d)	(– 6, 0, 0)	$2x - 3y + 6z - 2 = 0$

Miscellaneous Examples

Example 26 A line makes angles α, β, γ and δ with the diagonals of a cube, prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$

Solution A cube is a rectangular parallelopiped having equal length, breadth and height. Let OADBFEGC be the cube with each side of length a units. (Fig 11.21)

The four diagonals are OE, AF, BG and CD.

The direction cosines of the diagonal OE which is the line joining two points O and E are

$$\frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}$$

i.e., $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

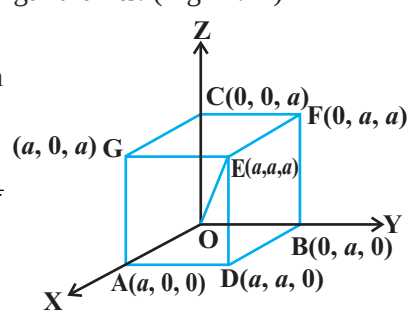

Fig 11.21

Similarly, the direction cosines of AF, BG and CD are $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$; $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$, respectively.

Let l, m, n be the direction cosines of the given line which makes angles $\alpha, \beta, \gamma, \delta$ with OE, AF, BG, CD, respectively. Then

$$\cos\alpha = \frac{1}{\sqrt{3}}(l + m + n); \cos\beta = \frac{1}{\sqrt{3}}(-l + m + n);$$

$$\cos\gamma = \frac{1}{\sqrt{3}}(l - m + n); \cos\delta = \frac{1}{\sqrt{3}}(l + m - n) \qquad \text{(Why?)}$$

Squaring and adding, we get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$

$$= \frac{1}{3}\left[(l + m + n)^2 + (-l + m + n)^2\right] + (l - m + n)^2 + (l + m - n)^2]$$

$$= \frac{1}{3}\left[4(l^2 + m^2 + n^2)\right] = \frac{4}{3} \qquad (\text{as } l^2 + m^2 + n^2 = 1)$$

Example 27 Find the equation of the plane that contains the point (1, – 1, 2) and is perpendicular to each of the planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$.

Solution The equation of the plane containing the given point is

$$A(x - 1) + B(y + 1) + C(z - 2) = 0 \qquad \ldots (1)$$

Applying the condition of perpendicularly to the plane given in (1) with the planes

$2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$, we have

$2A + 3B - 2C = 0$ and $A + 2B - 3C = 0$

Solving these equations, we find $A = -5C$ and $B = 4C$. Hence, the required equation is

$$-5C(x - 1) + 4C(y + 1) + C(z - 2) = 0$$

i.e. $\quad 5x - 4y - z = 7$

Example 28 Find the distance between the point P(6, 5, 9) and the plane determined by the points A (3, – 1, 2), B (5, 2, 4) and C(– 1, – 1, 6).

Solution Let A, B, C be the three points in the plane. D is the foot of the perpendicular drawn from a point P to the plane. PD is the required distance to be determined, which is the projection of $\overrightarrow{AP}$ on $\overrightarrow{AB} \times \overrightarrow{AC}$.

Hence, PD = the dot product of $\overrightarrow{AP}$ with the unit vector along $\overrightarrow{AB} \times \overrightarrow{AC}$.

So $$\overrightarrow{AP} = 3\hat{i} + 6\hat{j} + 7\hat{k}$$

and $$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix} = 12\hat{i} - 16\hat{j} + 12\hat{k}$$

Unit vector along $$\overrightarrow{AB} \times \overrightarrow{AC} = \frac{3\hat{i} - 4\hat{j} + 3\hat{k}}{\sqrt{34}}$$

Hence $$PD = (3\hat{i} + 6\hat{j} + 7\hat{k}) . \frac{3\hat{i} - 4\hat{j} + 3\hat{k}}{\sqrt{34}}$$

$$= \frac{3\sqrt{34}}{17}$$

Alternatively, find the equation of the plane passing through A, B and C and then compute the distance of the point P from the plane.

Example 29 Show that the lines

$$\frac{x - a + d}{\alpha - \delta} = \frac{y - a}{\alpha} = \frac{z - a - d}{\alpha + \delta}$$

and $$\frac{x - b + c}{\beta - \gamma} = \frac{y - b}{\beta} = \frac{z - b - c}{\beta + \gamma}$$ are coplanar.

Solution

Here

$$x_1 = a - d \qquad x_2 = b - c$$
$$y_1 = a \qquad y_2 = b$$
$$z_1 = a + d \qquad z_2 = b + c$$
$$a_1 = \alpha - \delta \qquad a_2 = \beta - \gamma$$
$$b_1 = \alpha \qquad b_2 = \beta$$
$$c_1 = \alpha + \delta \qquad c_2 = \beta + \gamma$$

Now consider the determinant

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} b - c - a + d & b - a & b + c - a - d \\ \alpha - \delta & \alpha & \alpha + \delta \\ \beta - \gamma & \beta & \beta + \gamma \end{vmatrix}$$

Adding third column to the first column, we get

$$2\begin{vmatrix} b-a & b-a & b+c-a-d \\ \alpha & \alpha & \alpha+\delta \\ \beta & \beta & \beta+\gamma \end{vmatrix} = 0$$

Since the first and second columns are identical. Hence, the given two lines are coplanar.

Example 30 Find the coordinates of the point where the line through the points A (3, 4, 1) and B (5, 1, 6) crosses the XY-plane.

Solution The vector equation of the line through the points A and B is

$$\vec{r} = 3\hat{i} + 4\hat{j} + \hat{k} + \lambda[(5-3)\hat{i} + (1-4)\hat{j} + (6-1)\hat{k}]$$

i.e. $$\vec{r} = 3\hat{i} + 4\hat{j} + \hat{k} + \lambda(2\hat{i} - 3\hat{j} + 5\hat{k}) \quad \text{... (1)}$$

Let P be the point where the line AB crosses the XY-plane. Then the position vector of the point P is of the form $x\hat{i} + y\hat{j}$.

This point must satisfy the equation (1). (Why ?)

i.e. $$x\hat{i} + y\hat{j} = (3+2\lambda)\hat{i} + (4-3\lambda)\hat{j} + (1+5\lambda)\hat{k}$$

Equating the like coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we have

$$x = 3 + 2\lambda$$
$$y = 4 - 3\lambda$$
$$0 = 1 + 5\lambda$$

Solving the above equations, we get

$$x = \frac{13}{5} \text{ and } y = \frac{23}{5}$$

Hence, the coordinates of the required point are $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$.

Miscellaneous Exercise on Chapter 11

1. Show that the line joining the origin to the point (2, 1, 1) is perpendicular to the line determined by the points (3, 5, – 1), (4, 3, – 1).

2. If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both of these are $m_1 n_2 - m_2 n_1$, $n_1 l_2 - n_2 l_1$, $l_1 m_2 - l_2 m_1$

3. Find the angle between the lines whose direction ratios are a, b, c and $b - c$, $c - a$, $a - b$.

4. Find the equation of a line parallel to x-axis and passing through the origin.

5. If the coordinates of the points A, B, C, D be (1, 2, 3), (4, 5, 7), (– 4, 3, – 6) and (2, 9, 2) respectively, then find the angle between the lines AB and CD.

6. If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are perpendicular, find the value of k.

7. Find the vector equation of the line passing through (1, 2, 3) and perpendicular to the plane $\vec{r} \cdot (\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$.

8. Find the equation of the plane passing through (a, b, c) and parallel to the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$.

9. Find the shortest distance between lines $\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$ and $\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$.

10. Find the coordinates of the point where the line through (5, 1, 6) and (3, 4,1) crosses the YZ-plane.

11. Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses the ZX-plane.

12. Find the coordinates of the point where the line through (3, – 4, – 5) and (2, – 3, 1) crosses the plane $2x + y + z = 7$.

13. Find the equation of the plane passing through the point (– 1, 3, 2) and perpendicular to each of the planes $x + 2y + 3z = 5$ and $3x + 3y + z = 0$.

14. If the points (1, 1, p) and (– 3, 0, 1) be equidistant from the plane $\vec{r} \cdot (3\hat{i} + 4\hat{j} - 12\hat{k}) + 13 = 0$, then find the value of p.

15. Find the equation of the plane passing through the line of intersection of the planes $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$ and $\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ and parallel to x-axis.

16. If O be the origin and the coordinates of P be (1, 2, – 3), then find the equation of the plane passing through P and perpendicular to OP.

17. Find the equation of the plane which contains the line of intersection of the planes $\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0$, $\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5 = 0$ and which is perpendicular to the plane $\vec{r} \cdot (5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0$.

18. Find the distance of the point (– 1, – 5, – 10) from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$.

19. Find the vector equation of the line passing through (1, 2, 3) and parallel to the planes $\vec{r} \cdot (\hat{i} - \hat{j} + 2\hat{k}) = 5$ and $\vec{r} \cdot (3\hat{i} + \hat{j} + \hat{k}) = 6$.

20. Find the vector equation of the line passing through the point (1, 2, – 4) and perpendicular to the two lines:

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \text{ and } \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}.$$

21. Prove that if a plane has the intercepts a, b, c and is at a distance of p units from the origin, then $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}$.

Choose the correct answer in Exercises 22 and 23.

22. Distance between the two planes: $2x + 3y + 4z = 4$ and $4x + 6y + 8z = 12$ is

(A) 2 units (B) 4 units (C) 8 units (D) $\frac{2}{\sqrt{29}}$ units

23. The planes: $2x - y + 4z = 5$ and $5x - 2.5y + 10z = 6$ are

(A) Perpendicular (B) Parallel

(C) intersect y-axis (D) passes through $\left(0, 0, \frac{5}{4}\right)$

Summary

- **Direction cosines of a line** are the cosines of the angles made by the line with the positive directions of the coordinate axes.
- If l, m, n are the direction cosines of a line, then $l^2 + m^2 + n^2 = 1$.
- Direction cosines of a line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are

  $$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$$

  where $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- **Direction ratios of a line** are the numbers which are proportional to the direction cosines of a line.
- If l, m, n are the direction cosines and a, b, c are the direction ratios of a line

then

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}\ ;\ m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}\ ;\ n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

- **Skew lines** are lines in space which are neither parallel nor intersecting. They lie in different planes.
- **Angle between skew lines** is the angle between two intersecting lines drawn from any point (preferably through the origin) parallel to each of the skew lines.
- If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two lines; and θ is the acute angle between the two lines; then

$$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$$

- If a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of two lines and θ is the acute angle between the two lines; then

$$\cos\theta = \left| \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

- Vector equation of a line that passes through the given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda \vec{b}$.
- Equation of a line through a point (x_1, y_1, z_1) and having direction cosines l, m, n is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

- The vector equation of a line which passes through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda (\vec{b} - \vec{a})$.
- Cartesian equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$.
- If θ is the acute angle between $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$, then

$$\cos\theta = \left| \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1|\,|\vec{b}_2|} \right|$$

- If $\dfrac{x - x_1}{l_1} = \dfrac{y - y_1}{m_1} = \dfrac{z - z_1}{n_1}$ and $\dfrac{x - x_2}{l_2} = \dfrac{y - y_2}{m_2} = \dfrac{z - z_2}{n_2}$ are the equations of two lines, then the acute angle between the two lines is given by $\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$.

- Shortest distance between two skew lines is the line segment perpendicular to both the lines.
- Shortest distance between $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ is

$$\left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

- Shortest distance between the lines: $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and

$\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ is

$$\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

- Distance between parallel lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}$ is

$$\left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right|$$

- In the vector form, equation of a plane which is at a distance d from the origin, and $\hat{n}$ is the unit vector normal to the plane through the origin is $\vec{r} \cdot \hat{n} = d$.
- Equation of a plane which is at a distance of d from the origin and the direction cosines of the normal to the plane as l, m, n is $lx + my + nz = d$.
- The equation of a plane through a point whose position vector is $\vec{a}$ and perpendicular to the vector $\vec{N}$ is $(\vec{r} - \vec{a}) . \vec{N} = 0$.
- Equation of a plane perpendicular to a given line with direction ratios A, B, C and passing through a given point (x_1, y_1, z_1) is

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

- Equation of a plane passing through three non collinear points (x_1, y_1, z_1),

(x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

- Vector equation of a plane that contains three non collinear points having position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is $(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$
- Equation of a plane that cuts the coordinates axes at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

- Vector equation of a plane that passes through the intersection of planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is $\vec{r} \cdot (\vec{n}_1 + \lambda \vec{n}_2) = d_1 + \lambda d_2$, where λ is any nonzero constant.
- Vector equation of a plane that passes through the intersection of two given planes $A_1 x + B_1 y + C_1 z + D_1 = 0$ and $A_2 x + B_2 y + C_2 z + D_2 = 0$ is $(A_1 x + B_1 y + C_1 z + D_1) + \lambda(A_2 x + B_2 y + C_2 z + D_2) = 0$.
- Two planes $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ are coplanar if

$$(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 0$$

- Two planes $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$ are coplanar if $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$
- In the vector form, if θ is the angle between the two planes, $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$, then $\theta = \cos^{-1} \dfrac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}$.
- The angle ϕ between the line $\vec{r} = \vec{a} + \lambda \vec{b}$ and the plane $\vec{r} \cdot \hat{n} = d$ is

$$\sin \phi = \left| \frac{\vec{b} \cdot \hat{n}}{|\vec{b}||\hat{n}|} \right|$$

- The angle θ between the planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is given by

$$\cos\theta = \left| \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}} \right|$$

- The distance of a point whose position vector is $\vec{a}$ from the plane $\vec{r}\cdot\hat{n} = d$ is $|d - \vec{a}\cdot\hat{n}|$
- The distance from a point (x_1, y_1, z_1) to the plane $Ax + By + Cz + D = 0$ is

$$\left| \frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}} \right|.$$